Sobolev II

$p > n = 2$. $|\Omega| = 1$.

$$\|u\|_{L^2} \le \left(\int |u_1| \int |u_2| \right)^{1/2}$$
$$\le \frac{1}{2} \int |u_1| + |u_2| \quad — ①$$
$$\le \frac{1}{\sqrt{2}} \|\nabla u\|_{L^1} \le \frac{1}{\sqrt{2}} \|\nabla u\|_{L^p}$$

Given any $\gamma > 1$,

$$\| |u|^\gamma \|_{L^2} \le \frac{\gamma}{\sqrt{2}} \| |u|^{\gamma-1} \|_{L^{p'}} \|\nabla u\|_{L^p} \quad — ②$$

where $p' = \frac{p}{p-1}$.

we define $\tilde{u} = \frac{\sqrt{2}\,|u|}{\|\nabla u\|_{L^p}}$

Note $\|\tilde{u}\|_{L^2} \le 1$ — ③ $(\because ①)$

By ②, we have

$$\|\tilde{u}^r\|_{L^2} \le \gamma \|\tilde{u}^{r-1}\|_{L^{p'}}.$$

$$\Rightarrow \|\tilde{u}\|_{L^{2r}} \le \gamma^{1/r} \|\tilde{u}\|^{1-1/r}_{L^{p'(r-1)}}$$

$$\le \gamma^{1/r} \|\tilde{u}\|^{1-1/r}_{L^{p'r}} \quad \text{④} \quad (\gamma'(|\Omega|=1))$$

Let $\beta = 2/p' > 1$, $\Rightarrow \beta^k > 1 \quad \forall k \in \mathbb{N}$.

$$\Rightarrow \|\tilde{u}\|_{L^{2\beta^k}} \le \beta^{k\beta^{-k}} \|\tilde{u}\|^{1-\beta^{-k}}_{L^{2\beta^{k-1}}} \quad \forall k \in \mathbb{N}. \quad \text{⑤}$$

($r = \beta^k$ at ④)

By ③, $\|\tilde{u}\|_{L^{2\beta}} \le \beta^{1/\beta}$.

by iterating ⑤, we have

$$\|\tilde{u}\|_{\beta^k} \le \beta^{\sum_{j=1}^{k} j\beta^{-j}} \le C(p).$$

By passing $k \to +\infty$, we obtain

$$\sup |\tilde{u}| \le C(p)$$

$$\therefore \sup |u| \le C \|\nabla u\|_{L^p}$$

Consider $y = |\Omega|^{\frac{1}{2}} x$.

$$\Rightarrow \sup |u| \le C |\Omega|^{\frac{1}{2} - \frac{1}{p}} \|\nabla u\|_{L^p}.$$

Morrey inequality. (if $p > n$).

$$\|u\|_{C^\alpha} \le C(n, p, \Omega) \|\nabla u\|_{L^p}$$

where $\alpha = 1 - \frac{n}{p} \in (0,1)$

Eigenfunction and Eigenvalue

1D wave on bounded interval

$u_{tt} = u_{xx}$ in $[0, L] \times [0, T)$

$u(0,t) = u(L,t)$ for $t \geq 0$

$\Rightarrow u(x,t) = \sum_{k=1}^{\infty} \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi x}{L}\right) a_k(t)$

where $a_k(t) = \int_0^L u(x,t) \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi x}{L}\right) dx$

$a_k'' = \int u_{tt} \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L} x\right) dx$

$= \int u_{xx} \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L} x\right) dx$

$= -\left(\frac{k\pi}{L}\right)^2 \int u \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L} x\right) dx$

$= -\left(\frac{k\pi}{L}\right)^2 a_k.$

$$a_k(t) = c_1 \sin\left(\frac{k\pi}{L} t\right) + c_2 \cos\left(\frac{k\pi}{L} t\right).$$

$$\therefore u(x,t) = \sum \alpha_k \sin\left(\frac{k\pi}{L} t\right) \underbrace{\sin\left(\frac{\pi k}{L} x\right)}_{\text{eigenfunction}}$$

$$+ \sum \beta_k \cos\left(\frac{k\pi}{L} t\right) \sin\left(\frac{k\pi}{L} x\right).$$

$$\underbrace{\frac{d^2}{dx^2}}_{\text{Linear operator}} \underbrace{\sin\left(\frac{k\pi}{L} x\right)}_{\text{eigenfunction}} = \underbrace{-\left(\frac{k\pi}{L}\right)^2}_{\text{eigenvalue}} \sin\left(\frac{k\pi}{L} x\right).$$

c.f. $\underbrace{A}_{\text{Linear map}} \underbrace{x}_{\text{eigenvector}} = \underbrace{\lambda}_{\text{eigenvalue}} x$

A is symmetric, $\Rightarrow$ A has eigenvectors $x_1, \ldots, x_n$ such that $\text{span}\{x_1, \ldots, x_n\} = \mathbb{R}^n$.

$\text{span}\{\sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \ldots\} = L^2[0,L]$

where $L^2([0,L]) = \{f : [0,L] \to \mathbb{R} \mid f^2 \text{ is integrable}\}$.

Fact) Let Ω be a bounded open set in $\mathbb{R}^n$ w/ smooth boundary.

Then, there exist a sequence $\{(w_i, \lambda_i)\}_{i=1}^{\infty}$ such that

$$\Delta w_i + \lambda_i w_i = 0 \quad \text{in } \Omega$$

$$w_i = 0 \quad \text{on } \partial\Omega$$

$$w_i \in C^{\infty}(\Omega) \cap C^0(\overline{\Omega}).$$

$$0 < \lambda_i \leq \lambda_{i+1}. \quad , \quad \|w_i\|_{L^2(\Omega)} = 1.$$

$$\lim_{i \to +\infty} \lambda_i = +\infty.$$

$$\{w_i\}_{i=1}^{\infty} \text{ spans } L^2(\Omega).$$

where $L^2(\Omega) = \{f : \Omega \to \mathbb{R} \mid f^2 \text{ is integrable}\}$.

(w_i are called the Dirichlet Laplace eigenfunction (w/ eigenvalue λ_i))

If $\partial_\nu w_i = 0$ on $\partial\Omega$ then w_i's are the Neumann Laplace eigenfunction

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Ex) Suppose $u \in C^\infty$.

$$\Delta u = u_{tt}$$

$$\Rightarrow u = \sum a_k(t)\, w_k(x)$$

$$\Rightarrow a_k = \int_\Omega u w_k$$

$$a_k'' = \int u_{tt} w_k = \int (\Delta u) w_k = \int u \Delta w_k$$

$$a_k'' = -\lambda_k \int u w_k = -\lambda_k a_k.$$

$$a_k = C_1 \sin(\sqrt{\lambda_k}\, t) + C_2 \cos(\sqrt{\lambda_k}\, t).$$

$$\therefore u(x,t) = \sum_{k=1}^{\infty} \alpha_k \sin(\sqrt{\lambda_k}\, t)\, w_k(x)$$
$$+ \sum \beta_k \cos(\sqrt{\lambda_k}\, t)\, w_k(x)$$

Existence

consider $u^m = \sum_{k=1}^{m} \alpha_k \sin(\sqrt{\lambda_k}\, t)\, w_k(x)$
$$+ \sum_{k=1}^{m} \beta_k \cos(\;"\;) \;\; " \;.$$

choose α_k, β_k by using g, h.

$\lim u^m(x,0) = g(x)$. $\lim u^m_t(x,0) = h(x)$

$u^m_{tt} = \Delta u^m$, $\quad u^m = 0$ on $\partial\Omega$.

Fact) Suppose $a_{ij} \in C^\infty(\bar{\Omega})$

$V \in C^\infty(\bar{\Omega})$

$\exists \{(w_i, \lambda_i)\}$ s.t. $w_i = 0$ on $\partial\Omega$

$$\partial_i(a_{ij}(x) w_j(x)) + (V(x) + \lambda_i) w_i(x) = 0$$

in Ω.

$\lambda_i \leq \lambda_{i+1}$ ($\lambda_1 < 0$ is possible)

$$\lim_{i \to +\infty} \lambda_i = +\infty$$

$\{w_i\}$ spans $L^2(\Omega)$

By using these eigenpairs

we can solve

$$\partial_i(a_{ij} u_j) + Vu = f. \quad + (u_t) \quad + (u_{tt})$$

Fredholm alternative

$$\Delta u(x) + V(x)u(x) = f(x) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

bounded.

Suppose $\{w_i, \lambda_i\}$ are eigenpairs

s.t. $\{w_i\}$ spans $L^2(\Omega)$

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k < 0 = \lambda_{k+1} = \cdots = \lambda_{k+N}$$

$$< \lambda_{k+N+1} \le \cdots$$

$$\text{Null} = \text{span}\{w_{k+1}, \cdots, w_{k+N}\}$$

$$= \left\{ \sum_{i=1}^{N} c_i w_{k+i} \,\middle|\, c_i \in \mathbb{R} \right\} \subset L^2(\Omega)$$

$$\text{proj}_N f = \sum_{i=1}^{N} \langle f, w_{k+i} \rangle w_{k+i}$$

$$\langle f, w_{k+i} \rangle = \int_\Omega f w_{k+i} \, dx$$

Case 1) $N = 0$. $Null = \phi$.

There exists a unique solution

$$u(x) = -\sum_{i=1}^{\infty} \frac{1}{\lambda_i} \langle f, w_i \rangle w_i.$$

where $\langle f, w_i \rangle = \int_\Omega f w_i \, dx$

Case 2) $Null \neq \phi$. $proj_N f = 0$.

there exist ∞ many solutions

$$u(x) = -\sum_{i=1}^{k} \lambda_i^{-1} \langle f, w_i \rangle w_i$$

$$- \sum_{i=1}^{\infty} \lambda_{i+k+N}^{-1} \langle f, w_{i+k+N} \rangle w_{i+k+N}$$

$$+ \sum_{i=1}^{N} c_i w_{i+k}$$

where $c_1, \cdots, c_N \in \mathbb{R}$.

This is because

$$\Delta u + \nu u = -\sum_{i\le k,\, i>k+N} \lambda_i^{-1} \langle f, w_i\rangle \underbrace{(\Delta w_i + \nu w_i)}_{=-\lambda_i w_i} + \sum_{i=1}^{N} c_i (\Delta w_{k+i} \cancel{+ \nu w_{k+i}}^{\,0})$$

$$= \sum_{i\le k,\, i>k+N} \langle f, w_i\rangle w_i$$

$$= f - \cancel{\mathrm{proj}_{V_N} f}^{\,0} = f$$

Case 3) $\mathrm{proj}_{V_N} f \neq 0$.

There is NO solution.

$$\left(\because \mathrm{proj}_{V_N} f = \sum_{i=k+1}^{k+N} \langle f, w_i\rangle w_i = \sum_{i=k+1}^{k+N} \langle \Delta u + \nu u, w_i\rangle \right.$$

$$\left. = \sum_{i=k+1}^{k+N} \langle \Delta w_i \cancel{+ \nu w_i}^{\,0}, u\rangle = 0 \right)$$

The first eigenfunction is an Energy minimizer !!

Let's consider the linear operator

$$Lu = \Delta u + Vu.$$

and eigenpairs (w_i, λ_i) i.e.

$$L w_i + \lambda_i w_i = 0 \quad \text{in } \Omega.$$
$$w_i = 0 \quad \text{on } \partial\Omega.$$

$$E(u) = \frac{\int_\Omega |\nabla u|^2 - V u^2 \, dx}{\int_\Omega u^2 \, dx}$$

where $\int u^2 \neq 0$.

$$E(u) \geq \frac{\int |\nabla u|^2 - (\sup V) \int u^2}{\int u^2}$$

$$= \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2} - \sup V$$

suppose $u = 0$ on $\partial\Omega$

$\Rightarrow$ $E(u) \geq C(n, \Omega) - \sup V$

By the Sobolev inequality

+ Holder "

$$\lambda_1 = \inf \left\{ E(u) \,\middle|\, \begin{array}{l} u = 0 \text{ on } \partial\Omega \\ u \in C^\infty(\bar{\Omega}) \end{array} \right\}$$

$(\lambda_1 \in \mathbb{R})$

Fact) λ_1 is the 1st eigenvalue !!

Suppose $\exists\, w_1 \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$

s.t. $w_1 = 0$ on $\partial\Omega$, $\int_\Omega w_1^2 \neq 0$

$$\lambda_1 = E(w_1) = \frac{\int |\nabla w_1|^2 - V w_1^2}{\int w_1^2}$$

Then, given $y \in \Omega$

we consider $\eta_\varepsilon(x-y) \in C^\infty(\Omega)$

$\eta_\varepsilon > 0$ in $B_\varepsilon(y)$. $\eta_\varepsilon = 0$ in $B_\varepsilon^c(y)$

$\overline{B_\varepsilon(y)} \subset \Omega$;

and define

$$I(s) = E(w_1 + s\eta_\varepsilon), \quad s \in \mathbb{R}$$

for small enough $|s|$. $\|w_1 + s\eta_\varepsilon\|_{L^2} \neq 0$

$w_1 + s\eta_\varepsilon = 0$ on $\partial\Omega$. $w_1 + s\eta_\varepsilon \in C^\infty$

$$I(s) \geq I(0) \Rightarrow I'(0) = 0$$

$$\frac{1}{2}\frac{d}{ds}\log I(s)$$

$$= \frac{\int \nabla(w_1 + s\eta_2)\cdot\nabla\eta_2 - V(w_1+s\eta_2)\eta_2}{\int |\nabla(w_1+s\eta_2)|^2 - V(w_1+s\eta_2)^2}$$

$$- \frac{\int (w_1 + s\eta_2)\eta_2}{\int (w_1+s\eta_2)^2}$$

$$0 = \frac{1}{2}\frac{I'(0)}{I(0)} = \frac{\int \nabla w_1 \cdot \nabla\eta_2 - Vw_1\eta_2}{\int |\nabla w_1|^2 - Vw_1^2}$$

$$- \int w_1\eta_2 \Big/ \int w_1^2$$

$$= \frac{\int -\eta_2\Delta w_1 - V\eta_2 w_1 - \lambda_1 w_1 \eta_2}{\int |\nabla w_1|^2 - Vw_1^2}$$

$\therefore\ 0 = \int (\Delta w_1 + V w_1 + \lambda_1 w_1)\eta_\varepsilon$

By choosing ε small enough

we have $\Delta w_1 + V w_1 + \lambda_1 w_1 = 0$

in a nbd of y.

$\Rightarrow L w_1 + \lambda_1 w_1 = 0$ in Ω.